SADLER MATHEMATICS METHODS UNIT 3

WORKED SOLUTIONS

Chapter 4 Area under a curve

Exercise 4A

Question 1

a By counting squares $\approx 19\frac{1}{2}$ squares

$$\text{Each square} = 0.1 \times 0.1$$
$$= 0.01 \text{ units}^2$$

$$19.5 \times 0.01 = 0.195$$

$\therefore$ Approximately 0.2 units^2

b Underestimate:

$$= 0.1 \times 0.5^4 + 0.1 \times 0.6^4 + 0.1 \times 0.7^4 + 0.1 \times 0.8^4 + 0.1 \times 0.9^4$$
$$= 0.1 \times 1.4979$$
$$= 0.14979$$

Overestimate:

$$= 0.1 \times 0.6^4 + 0.1 \times 0.7^4 + 0.1 \times 0.8^4 + 0.1 \times 0.9^4 + 0.1 \times 1^4$$
$$= 0.1 \times 2.4354$$
$$= 0.24354$$

Mean:

$$= \frac{0.14979 + 0.24354}{2}$$
$$= 0.196665$$

$\therefore$ Approximately 0.197

Question 2

a Estimate by counting squares $\approx 20\frac{1}{2}$ squares.

$$\text{Each square} = 0.2 \times 0.4$$
$$= 0.08 \text{ units}^2$$

$$20.5 \times 0.08 = 1.64$$

$\therefore$ Approximately 1.7 units^2.

b Underestimate:

$$= 0.2 \times (4 - 1.2)^2 + 0.2 \times (4 - 1.4^2) + 0.2 \times (4 - 1.6^2) + 0.2 \times (4 - 1.8^2)$$
$$= 0.2 \times 6.8$$
$$= 1.36 \text{ units}^2$$

Overestimate:

$$= 0.2 \times (4 - 1)^2 + 0.2 \times (4 - 1.2^2) + 0.2 \times (4 - 1.4^2) + 0.2 \times (4 - 1.6^2) + 0.2 \times (4 - 1.8^2)$$
$$= 1.96 \text{ units}^2$$

Mean:

$$= \frac{1.36 + 1.96}{2}$$
$$= 1.66 \text{ units}^2$$

Question 3

a Estimate by counting squares ≈ 33 squares.

$$\text{Each square} = 0.2 \times 1$$
$$= 0.2 \text{ units}^2$$

$$33 \times 0.2 = 6.6$$

$\therefore$ Approximately 6.6 units^2.

b Underestimate:

$$= 0.2 \times \left[0.2^2 + 2(0.2) + 0.4^2 + 2(0.4) + 0.6^2 + 2.(0.6) + ... + 1.8^2 + 2(1.8)\right]$$
$$= 0.2 \times 29.4$$
$$= 5.88 \text{ units}^2$$

Overestimate:

$$= 0.2 \times \left[0.2^2 + 2(0.2) + 0.4^2 + 2(0.4) + ... + 2^2 + 2(2)\right]$$
$$= 7.48 \text{ units}^2$$

Mean:

$$= \frac{5.88 + 7.48}{2}$$
$$= 6.68 \text{ units}^2$$

Exercise 4B

Question 1

a At $x = 5$,

$$y = 2(5) + 5$$
$$= 15$$

At $x = 9$,

$$y = 2(9) + 5$$
$$= 23$$

$$\text{Area} = \frac{1}{2} \times 4(15 + 23)$$
$$= 76 \text{ units}^2$$

b $$\int_5^9 (2x + 5)dx$$
$$= \left[x^2 + 5x\right]_5^9$$
$$= (81 + 45) - (25 + 25)$$
$$= 126 - 50$$
$$= 76 \text{ units}^2$$

Question 2

a At $x = 4$,

$$2y + 4 = 16$$
$$y = 6$$

At $x = 10$,

$$2y + 10 = 16$$
$$y = 3$$

$$\text{Area} = \frac{1}{2} \times 6(6 + 3)$$
$$= 27 \text{ units}^2$$

b

$$2y + x = 16$$
$$2y = 16 - x$$
$$y = 8 - \frac{1}{2}x$$

$$\int_4^{10} \left(8 - \frac{1}{2}x\right) dx$$

$$= \left[8x - \frac{x^2}{4}\right]_4^{10}$$
$$= (80 - 25) - (32 - 4)$$
$$= 27 \text{ units}^2$$

Question 3

$$\int_1^4 3\sqrt{x}\, dx$$
$$= \left[3 \times \frac{2}{3} x^{\frac{3}{2}}\right]_1^4$$
$$= \left(2 \times 4^{\frac{3}{2}} - 2 \times 1\right)$$
$$= 14$$

Question 4

a

$$\int_1^2 4x^{-2}\ dx$$
$$= \left[-\frac{4}{x}\right]_1^2$$
$$= (-2) - (-4)$$
$$= 2$$

Question 5

$$\int_0^2 \frac{x^2}{4}\ dx$$
$$= \left[\frac{x^3}{12}\right]_0^2$$
$$= \frac{8}{12}$$
$$= \frac{2}{3}$$

Question 6

$$\int_2^4 \frac{x^2}{4}\ dx$$
$$= \left[\frac{x^3}{12}\right]_2^4$$
$$= \left(\frac{4^3}{12} - \frac{2^3}{12}\right)$$
$$= 4\frac{2}{3}$$

Question 7

$$\int_1^3 10x\ dx$$
$$= \left[5x^2\right]_1^3$$
$$= \left(5 \times 3^2 - 5 \times 1^2\right)$$
$$= 40$$

Question 8

$$\int_{-1}^{1}(4x+5)\,dx$$
$$=\left[2x^2+5x\right]_{-1}^{1}$$
$$=(2+5)-(2-5)$$
$$=7-(-3)$$
$$=10$$

Question 9

$$\int_{2}^{2}(4-x^2)\,dx$$
$$=0$$

Question 10

$$\int_{2}^{3}3x^2\,dx$$
$$=\left[x^3\right]_{2}^{3}$$
$$=27-8$$
$$=19$$

Question 11

$$\int_{-1}^{2}(6x^2+7)\,dx$$
$$=\left[2x^3+7x\right]_{-1}^{2}$$
$$=\left(2\times 2^3+14\right)-\left(2(-1)^3-7\right)$$
$$=30-(-9)$$
$$=39$$

Question 12

$$\int_{0}^{3}(1+x^2)\,dx$$
$$=\left[x+\frac{x^3}{3}\right]_{0}^{3}$$
$$=\left(3+\frac{3^3}{3}\right)-0$$
$$=12$$

Question 13

$$\int_3^6 (x+x^2)\,dx$$

$$=\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_3^6$$

$$=\left(\frac{6^2}{2}+\frac{6^3}{3}\right)-\left(\frac{3^2}{2}+\frac{3^3}{3}\right)$$

$$=76.5$$

Question 14

$$\int_2^3 (9-x^2)\,dx$$

$$=\left[9x-\frac{x^3}{3}\right]_2^3$$

$$=(9\times 3-9)-\left(9\times 2-\frac{2^3}{3}\right)$$

$$=2\frac{2}{3}$$

Question 15

$$\int_0^1 (2+x)^4\,dx$$

$$=\left[\frac{(2+x)^5}{5}\right]_0^1$$

$$=\frac{3^5}{5}-\frac{2^5}{5}$$

$$=48.6-6.4$$

$$=42.2$$

Question 16

$$\int_0^1 (2+5x)^4\ dx$$
$$= \frac{1}{5}\int_0^1 5(2+5x)^4\ dx$$
$$= \frac{1}{5}\left[\frac{(2+5x)^5}{5}\right]_0^1$$
$$= \frac{1}{5}\left(\frac{7^5}{5} - \frac{2^5}{5}\right)$$
$$= 671$$

Question 17

$$\int_0^1 12x(1+x^2)^2 dx$$
$$= 6\int_0^1 2x(1+x^2)^2 dx$$
$$= 6\left[\frac{(1+x^2)^3}{3}\right]_0^1$$
$$= 6\left(\frac{(1+1)^3}{3} - \frac{(1+0)^3}{3}\right)$$
$$= 14$$

Question 18

$$\int_3^3 (4+x^2)^2\ dx$$
$$= 0$$

Question 19

$$\int_{-1}^{1}(1+x^2)^2\ dx$$

$$=\int_{-1}^{1}(1+2x^2+x^4)\ dx$$

$$=\left[\frac{x^5}{5}+\frac{2}{3}x^3+x\right]_{-1}^{1}$$

$$=\left(\frac{1}{5}+\frac{2}{3}+1\right)-\left(-\frac{1}{5}-\frac{2}{3}-1\right)$$

$$=1\frac{13}{15}-\left(-1\frac{13}{15}\right)$$

$$=3\frac{11}{15}$$

Question 20

a

$$\int_{0}^{1}(x^2)\ dx$$

$$=\left[\frac{x^3}{3}\right]_{0}^{1}$$

$$=\frac{1^3}{3}-0$$

$$=\frac{1}{3}$$

b

$$\int_{1}^{3}(x^2)\ dx$$

$$=\left[\frac{x^3}{3}\right]_{1}^{3}$$

$$=\frac{3^3}{3}-\frac{1^3}{3}$$

$$=8\frac{2}{3}$$

c

$$\int_{0}^{3}(x^2)\ dx$$

$$=\left[\frac{x^3}{3}\right]_{0}^{3}$$

$$=\frac{3^3}{3}-0$$

$$=9$$

Question 21

a $\int_0^4 (4x - x^2)\, dx$

$= \left[2x^2 - \frac{x^3}{3}\right]_0^4$

$= \left(2 \times 4^2 - \frac{4^3}{3}\right) - 0$

$= 10\frac{2}{3}$

b $\int_4^5 (4x - x^2)\, dx$

$= \left[2x^2 - \frac{x^3}{3}\right]_4^5$

$= \left(2 \times 5^2 - \frac{5^3}{3}\right) - \left(2 \times 4^2 - \frac{4^3}{3}\right)$

$= 8\frac{1}{3} - 10\frac{2}{3}$

$= -2\frac{1}{3}$

c $\int_0^5 (4x - x^2)\, dx$

$= \left[2x^2 - \frac{x^3}{3}\right]_0^5$

$= \left(2 \times 5^2 - \frac{5^3}{3}\right) - 0$

$= 8\frac{1}{3}$

Question 22

a $\int_1^3 (3x^2 + 2x)\, dx$

$= \left[x^3 + x^2 \right]_1^3$

$= (3^3 + 3^2) - (1 + 1)$

$= 34$

b $\int_3^1 (3x^2 + 2x)\, dx$

$= \left[x^3 + x^2 \right]_3^1$

$= (1 + 1) - (3^3 + 3^2)$

$= -34$

Question 23

a $\int_0^3 (x^2)\, dx$

$= \left[\frac{x^3}{3} \right]_0^3$

$= \frac{3^3}{3} - 0$

$= 9$

b $\int_0^3 (3x^2)\, dx$

$= \left[x^3 \right]_0^3$

$= 27$

c $\int_0^3 (4x^2)\, dx$

$= \left[\frac{4x^3}{3} \right]_0^3$

$= \left(\frac{4 \times 3^3}{3} \right) - 0$

$= 36$

Question 24

$$\int_{-\pi}^{\pi} (2x+3)\, dx$$
$$= \left[x^2 + 3x \right]_{-\pi}^{\pi}$$
$$= \left(\pi^2 + 3\pi \right) - \left(\pi^2 - 3\pi \right)$$
$$= 6\pi$$

Question 25

$$\int_{\sqrt{2}}^{2} (2x+6x^2)\, dx$$
$$= \left[x^2 + 2x^3 \right]_{\sqrt{2}}^{2}$$
$$= \left(2^2 + 2 \times 2^3 \right) - \left(\sqrt{2}^2 + 2(\sqrt{2})^2 \right)$$
$$= 18 - 4\sqrt{2}$$

Exercise 4C

Question 1

$a,\ e,\ f$

Question 2

a No

b No

c Yes

d No

e Yes

f Yes

g Yes

Question 3

$$\begin{aligned}&\int_0^2 (2x+1)\ dx\\&=\left[x^2+x\right]_0^2\\&=(4+2)-0\\&=6\end{aligned}$$

At $x=0$,

$$\begin{aligned}y&=2(0)+1\\&=1\end{aligned}$$

At $x=2$,

$$\begin{aligned}y&=2(2)+1\\&=5\end{aligned}$$

$$\begin{aligned}\text{Area}&=\frac{1}{2}(1+5)\times 2\\&=6\text{ units}^2\end{aligned}$$

Question 4

$$\int_2^4 \frac{x^4}{4}\ dx$$

$$=\left[\frac{x^5}{20}\right]_2^4$$

$$=\frac{4^5}{20}-\frac{2^5}{20}$$

$$=49.6 \text{ units}^2$$

Question 5

$$\int_1^3 \left((x-2)^2+3\right)\ dx$$

$$=\left[\frac{x-2)^3}{3}+3x\right]_1^3$$

$$=\left(\frac{1}{3}+9\right)-\left(-\frac{1}{3}+3\right)$$

$$=6\frac{2}{3} \text{ units}^2$$

Question 6

$$\begin{aligned} 8-2x^2 &= 0 \\ 2x^2 &= 8 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

$$\int_{-2}^2 (8-2x^2)\ dx$$

$$=\left[8x-\frac{2x^3}{3}\right]_{-2}^2$$

$$=-\left(16-\frac{2\times 8}{3}\right)-\left(-16+\frac{2\times 8}{3}\right)$$

$$=21\frac{1}{3} \text{ unit}^2$$

Question 7

$$\int_0^1 (1-x^3)\ dx$$

$$= \left[x - \frac{x^4}{4} \right]_0^1$$

$$= \left(1 - \frac{1}{4} \right) - 0$$

$$= \frac{3}{4} \text{ units}^2$$

Question 8

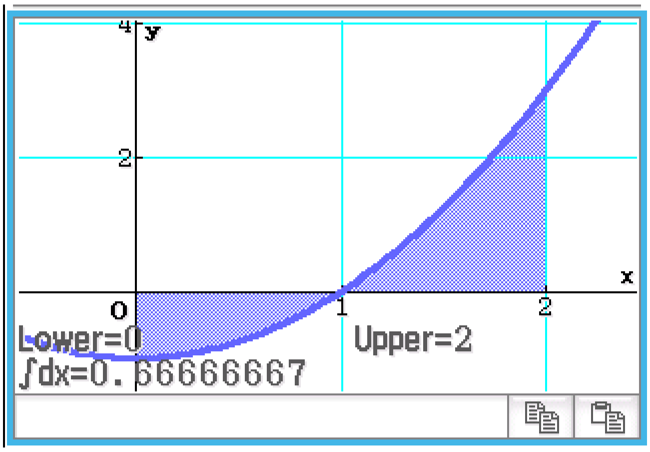
$$\int_{-2}^0 \left((x+1)^3 + 1 \right) dx$$

$$= \left[\frac{(x+1)^4}{4} + x \right]_{-2}^0$$

$$= \left(\frac{1}{4} + 0 \right) - \left(\frac{1}{4} + (-2) \right)$$

$$= 2 \text{ units}^2$$

Question 9



$\int_0^2 (x^2-1)\ dx$

$\int_0^1 (x^2-1)\ dx$

$=\left[\frac{x^3}{3}-x\right]_0^1$

$=\left(\frac{1}{3}-1\right)-0$

$=-\frac{2}{3}$

$\int_1^2 (x^2-1)\ dx$

$=\left[\frac{x^3}{3}-x\right]_1^2$

$=\left(\frac{8}{3}-2\right)-\left(\frac{1}{3}-1\right)$

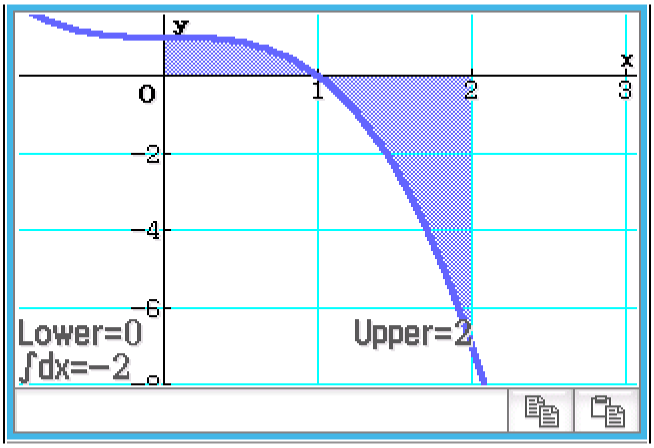
$=\frac{2}{3}-\left(-\frac{2}{3}\right)$

$=\frac{4}{3}$

$\therefore \text{Area}=\frac{4}{3}+\frac{2}{3}$

$=2\text{ units}^2$

Question 10



$\int_0^1 (1-x^3)\ dx-\int_1^2 (1-x^3)\ dx$

$=\left[x-\frac{x^4}{4}\right]_0^1-\left[x-\frac{x^4}{4}\right]_1^2$

$=\left(\left(1-\frac{1}{4}\right)-0\right)-\left(\left(2-\frac{16}{4}\right)-\left(1-\frac{1}{4}\right)\right)$

$=\frac{3}{4}-\left(-\frac{11}{4}\right)$

$=3.5\text{ units}^2$

Question 11

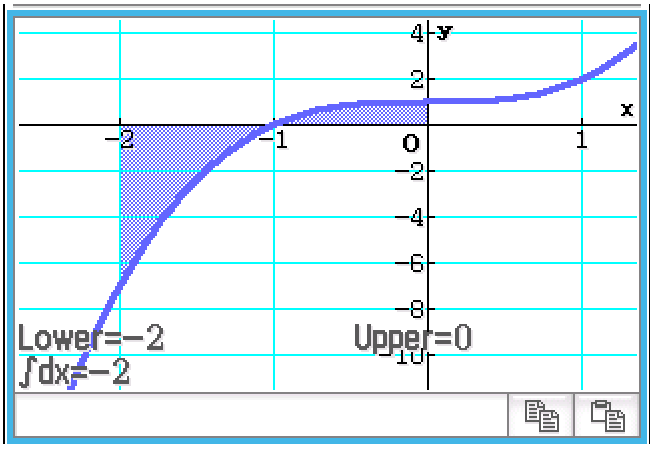
$$-\int_{-2}^{-1}(x+1)^3\ dx+\int_{-1}^{0}(x+1)^3\ dx$$

$$=-\left[\frac{(x+1)^4}{4}\right]_{-2}^{-1}+\left[\frac{(x+1)^4}{4}\right]_{-1}^{0}$$

$$=-\left(0-\frac{1}{4}\right)+\left(\frac{1}{4}-0\right)$$

$$=\frac{1}{4}+\frac{1}{4}$$

$$=0.5\text{ units}^2$$



Question 12

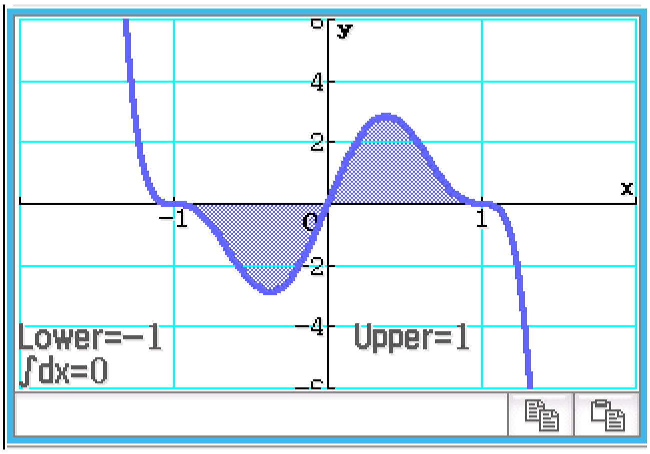
$$-\int_{-1}^{0}\left(12x(1-x^2)^3\right)\ dx+\int_{0}^{1}\left(12x(1-x^2)^3\right)\ dx$$

$$=-\left[-\frac{3}{2}(1-x^2)^4\right]_{1}^{0}+\left[-\frac{3}{2}(1-x^2)^4\right]_{0}^{1}$$

$$=-\left(-\frac{3}{2}\times1-\left(-\frac{3}{2}\times0\right)\right)+\left(-\frac{3}{2}\times0-\left(-\frac{3}{2}\times1\right)\right)$$

$$=\left(-\frac{3}{2}\right)+\left(\frac{3}{2}\right)$$

$$=3\text{ units}^2$$



Question 13

x-int :

$2 - x^2 = 0$

$x = \pm\sqrt{2}$

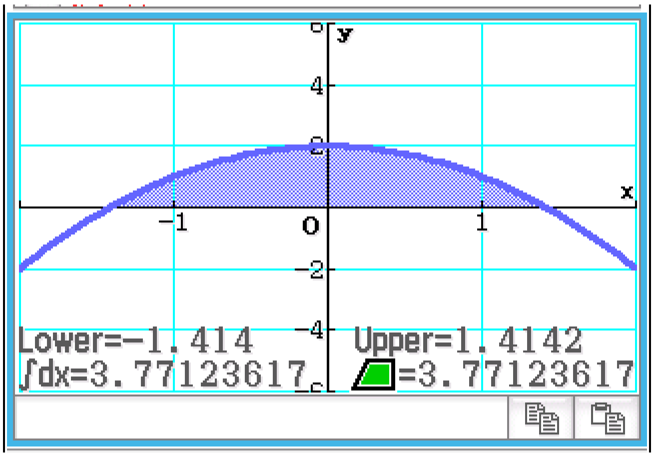
$\int_{-\sqrt{2}}^{\sqrt{2}} \left(2 - x^2\right) dx$

$= \left[2x - \frac{x^3}{3}\right]_{-\sqrt{2}}^{\sqrt{2}}$

$= \left(2\sqrt{2} - \frac{\sqrt{2}^3}{2}\right) - \left(2(-\sqrt{2}) - \frac{(-\sqrt{2})^3}{3}\right)$

$= \frac{4\sqrt{2}}{3} - \left(-\frac{4\sqrt{2}}{3}\right)$

$= \frac{8\sqrt{2}}{3}$ units^2



Question 14

$y = x^2$

x-int and y-int $(0, 0)$

$y = 3 - 2x$

y-int : $(0, 3)$

x-int : $(1.5, 0)$

Points of intersection :

$x^2 = 3 - 2x$

By ClassPad, $x = -3, 1$

Intersection points $(-3, 9)$ and $(1, 1)$

$\because \int_{-3}^{1} \left(3 - 2x - x^2\right) dx$

$= 10\frac{2}{3}$ units^2

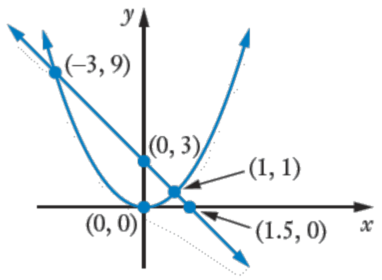
$(x-3)^2 = x - 1$

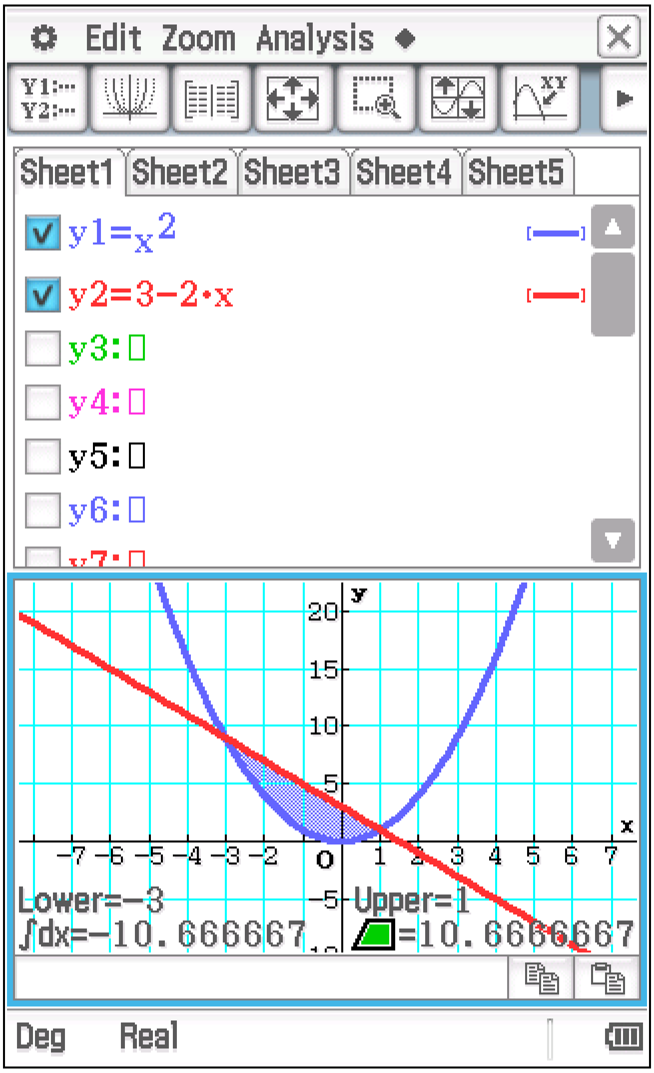
$\frac{d}{dx}(x-3)^2 = 2(x-3)$

$0 = 2(x-3)$

$x = 3$

$\therefore (3, 0)$ is a minimum turning point.



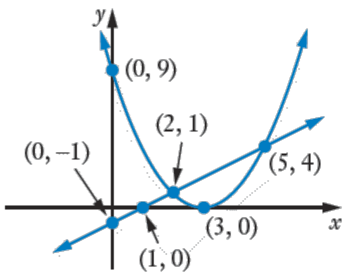


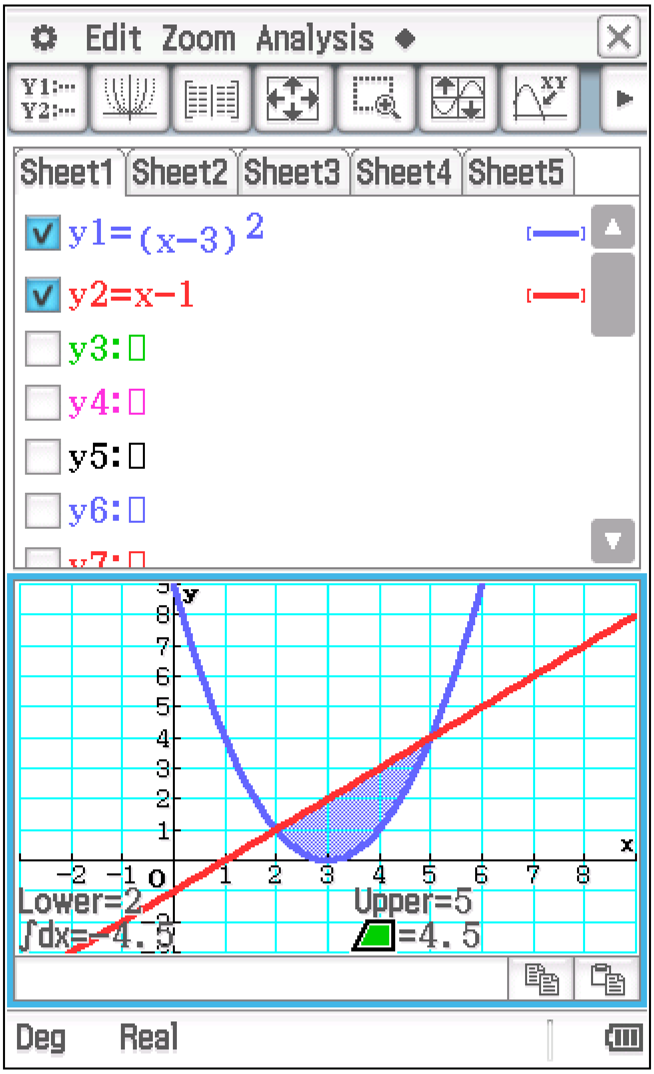
Question 15

By ClassPad, $x = 2, 5$

$\int_2^5 \left[(x-1)-(x-3)^2\right] dx$

$= 4.5 \text{ units}^2$





Question 16

$\frac{d}{dx}=(x^2-2x+3)=2x-2$

$0=2x-2$

$x=1$

At $x=1$, $x^2-2x+3=2$

$\therefore$ (1, 2) is a minimum turning point.

$\frac{d}{dx}(2x^2)=4x$

$0=4x$

$x=0$

At $x=0$, $2x^2=0$

$\therefore$ (0, 0) is a minimum turning point.

Points of Intersection :

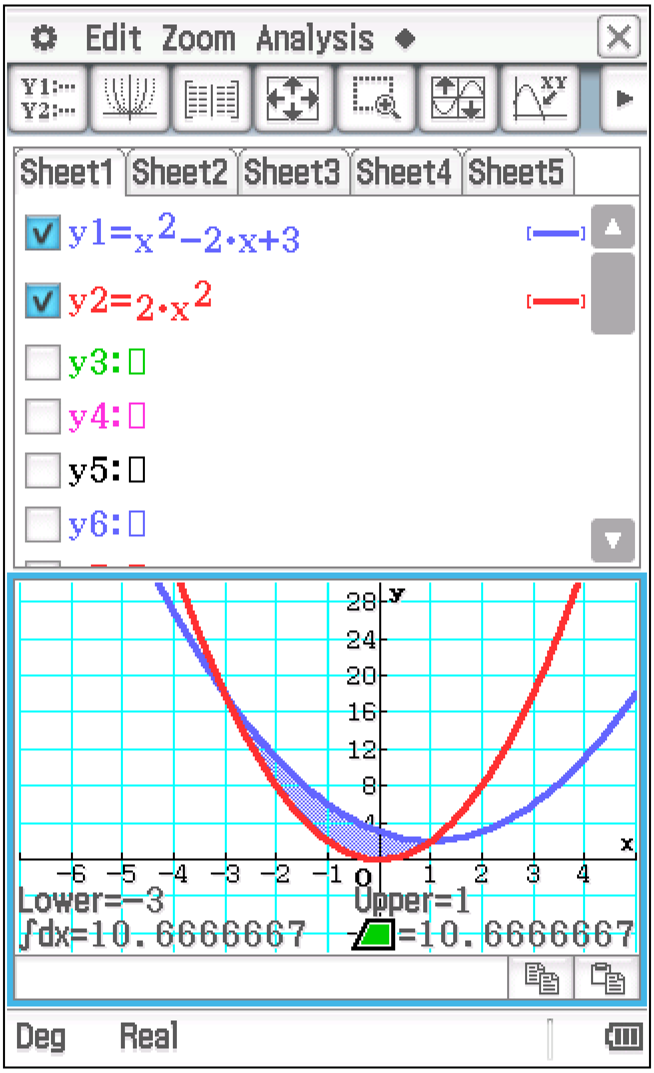
$x^2-2x+3=2x^2$

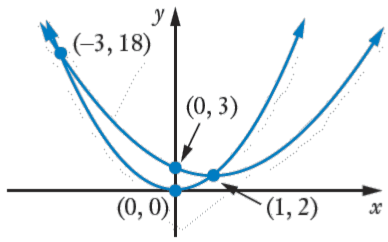
By ClassPad, $x=-3, 1$

$\int_{-3}^{1}\left[\left(x^2-2x+3\right)-\left(2x^2\right)\right]\,dx$

$=\int_{-3}^{1}\left(-x^2-2x+3\right)\,dx$

$=10\frac{2}{3}$ units^2





Question 17

$y = x$ has no stationary points.

$\frac{d}{dx}(x^3) = 3x^2$

$0 = 3x^2$

$x = 0$

At $x = 0,\ x^3 = 0$

$\therefore$ (0, 0) is a stationary point)

$\frac{d}{dx}(3x^2) = 6x$

At $x = 0,\ 6x = 0$

	-0.1	$x = 0$	0.1
$3x^2$	$+ve$	0	$+ve$
	/	–	/

$\therefore$ (0, 0) is a horizontal point of inflection.

$x^3 = x$

$x^3 - x = 0$

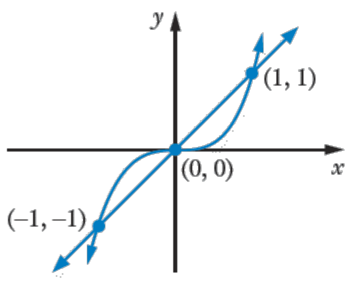
$x\left(x^2 - 1\right) = 0$

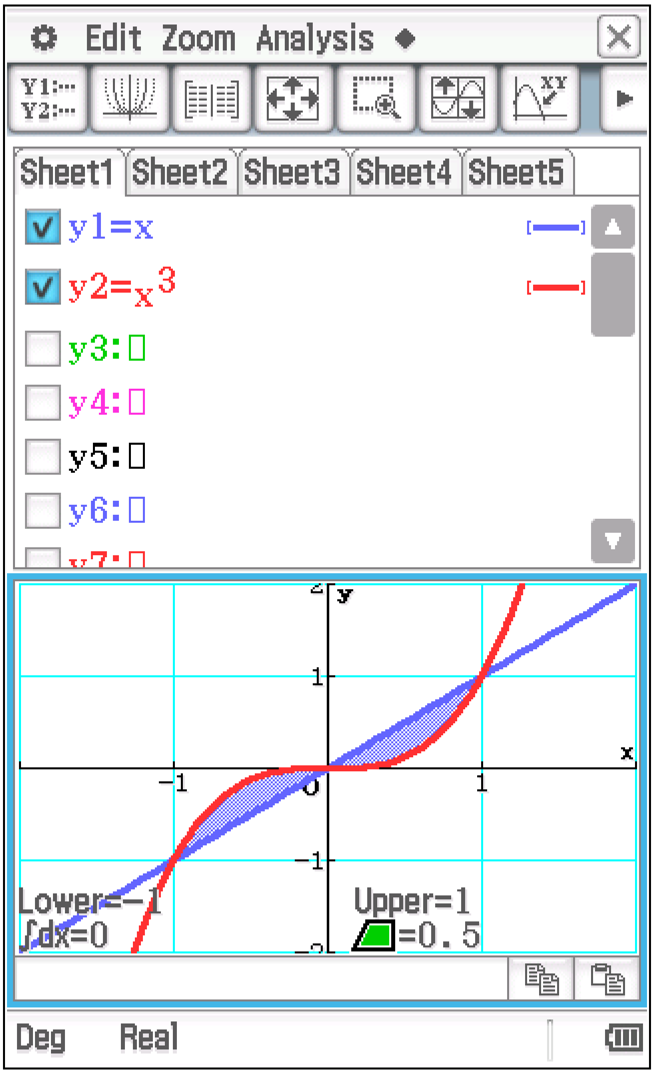
$x = -1,\ 0,\ 1$

$\int_{-1}^{0}\left(x^3 - x\right)\,dx + \int_{0}^{1}\left(x - x^3\right)\,dx$

$= \frac{1}{4} + \frac{1}{4}$

$= 0.5\ \text{units}^2$





Question 18

$y = 2x^3 - 3x$

$y\text{-int}: (0, 0)$

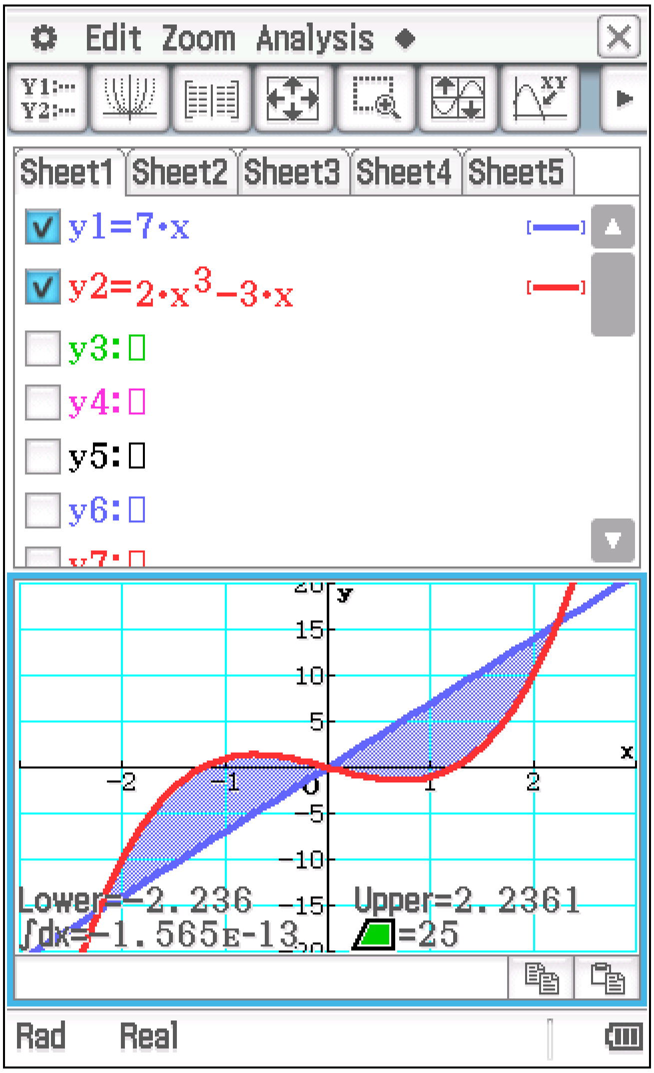
$x\text{-int}: 2x^3 - 3x = 0$

By ClassPad, $x = -\sqrt{\frac{3}{2}}, 0, \sqrt{\frac{3}{2}}$

Points of intersection:

$2x^3 - 3x = 7x$

By ClassPad, $x = -\sqrt{5}, 0, 5$



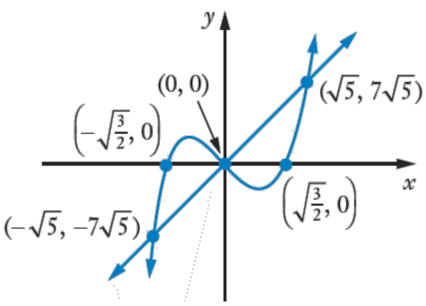
$$\int_{-\sqrt{5}}^{0}\left(2x^3 - 3x - 7x\right)\, dx + \int_{0}^{\sqrt{5}}\left[7x - \left(2x^3 - 3x\right)\right]\, dx$$

$$= \int_{-\sqrt{5}}^{0}\left(2x^3 - 10x\right)\, dx + \int_{0}^{\sqrt{5}}\left(10x - 2x^3\right)\, dx$$

$$= \left[\frac{x^4}{2} - 5x^2\right]_{-\sqrt{5}}^{0} + \left[2x^2 - \frac{x^4}{2}\right]_{0}^{\sqrt{5}}$$

$$= 12.5 + 12.5$$

$$= 25 \text{ units}^2$$



Question 19

$$\int_0^{10} \frac{\left(200 - x^2\right)}{50}\, dx$$
$$= \frac{100}{3}$$
$$\therefore \text{Area} = 2 \times \frac{100}{3}$$
$$= 66\frac{2}{3} \text{ units}^2$$

$$\text{Cost}: 66\frac{2}{3} \times 45$$
$$= \$3000$$

Exercise 4D

Question 1

a $\int_{10}^{20}\left(3x^2-60x+500\right)\,dx$

$=3000$

\$3000

b $\int_{40}^{50}\left(3x^2-60x+500\right)\,dx$

$=39\ 000$

\$39 000

Question 2

$\int_{25}^{100}\frac{250}{\sqrt{x}}\,dx$

$=2500$

\$2500

Question 3

a $\int_{10}^{20}\frac{400}{x+1}\,dx$

$=258.6509$

$\Rightarrow$ \$259

b $\int_{20}^{40}\frac{400}{x+1}\,dx$

$=267.6199$

$\Rightarrow$ \$268

Question 4

a $\int_5^8 40(25-t)\ dt$

b

$$\int_5^8 40(25-t)\,dt$$
$$=40\left[25t-\frac{t^2}{2}\right]_5^8$$
$$=40\left[25\times 8-\frac{8^2}{2}-\left(25\times 5-\frac{5^2}{2}\right)\right]$$
$$=40[200-32-125+12.5]$$
$$=40\times 55.5$$
$$=2220$$

Question 5

a $\int_5^{10} \frac{t^{0.1}}{2}\,dt$

b

$$\int_5^{10} \frac{t^{0.1}}{2}\,dt=0.5\left[\frac{t^{1.1}}{1.1}\right]_5^{10}$$
$$=\frac{0.5}{1.1}\left[10^{1.1}-5^{1.1}\right]$$
$$=3.0528\ldots$$
$$\approx 3.1$$

Question 6

$$\int_{20}^{28}(5.1+0.04t)\ dt$$
$$=48.48$$
$$\Rightarrow 48.5 \text{ million}$$

Question 7

a $\int_0^{10}\left(20-0.15t^2\right)\,dt=150$

150 kL

b $\int_0^{1}\left(20-0.15t^2\right)\,dt=20$

20 kL

c $\int_9^{10}\left(20-0.15t^2\right)\,dt=6$

6 kL

Question 8

a $\int_0^{4}\left(600+\frac{600}{(t+1)^2}\right)dt$

$=2880$

2880 sales

b $\int_4^{5}\left(600+\frac{600}{(t+1)^2}\right)dt$

$=620$

620 sales

Question 9

a $\int_0^{4}\left(150-\frac{600}{(t+2)^2}\right)dt$

$=400$

b $\int_4^{8}\left(150-\frac{600}{(t+2)^2}\right)dt$

$=560$

c $\int_1^{2}\left(150-\frac{600}{(t+2)^2}\right)dt$

$=100$

d $\int_3^{4}\left(150-\frac{600}{(t+2)^2}\right)dt$

$=130$

Question 1

$A: y = 4$
$B: x = -5$
$C: y = 0.5x + 2$
$D: y = x + 3$
$E: y = 2x - 4$
$F: y = -x - 1$
$G: y = x - 6$
$H: y = -2x - 10$

Question 2

a $f'(x) = 2 - 9x^2$

b $f'(5) = 2 - 9(5)^2 \ = -223$

c $f''(x) = -18x$

d $f''(-5) = -18(-5) = 90$

Question 3

a
$$\int_1^3 (2x)\ dx$$
$$= \left[x^2\right]_1^3$$
$$= 9 - 1$$
$$= 8$$

b
$$\int_1^4 \left(\sqrt{x}\right)\ dx$$
$$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4$$
$$= \frac{2}{3} \times (2^2)^{\frac{3}{2}} - \frac{2}{3} \times 1^{\frac{3}{2}}$$
$$= \frac{16}{3} - \frac{2}{3}$$
$$= \frac{14}{3}$$

Question 4

$$\frac{dy}{dx} = (x+5)\times 1 + (x-3)\times 1$$
$$= x+5+x-3$$
$$= 2x+2$$

Question 5

$$\frac{dy}{dx} = (x+5)(-1) + (3-x)\times 1$$
$$= -x-5+3-x$$
$$= -2x-2$$

Question 6

$$\frac{dy}{dx} = (2x+1)\times 1 + (x+5)\times 2$$
$$= 2x+1+2x+10$$
$$= 4x+11$$

Question 7

$$\frac{dy}{dx} = (5-2x)\times 2 + (2x+1)(-2)$$
$$= 10-4x-4x-2$$
$$= 8-8x$$

Question 8

$$\frac{dy}{dx} = (x+1)^2 \times 2 + (2x+7)\times 2(x+1)$$
$$= 2(x+1)^2 + 2(x+1)(2x+7)$$
$$= 2(x+1)\left[x+1+2x+7\right]$$
$$= 2(x+1)(3x+8)$$

Question 9

$$\frac{dy}{dx} = (2x+5)^3 \times 5 + (5x+6)\times 3(2x+5)^2 \times 2$$
$$= (2x+5)^2\left[5(2x+5)+6(5x+6)\right]$$
$$= (2x+5)^2(40x+61)$$

Question 10

a $\frac{dy}{dx} = 2(3x+1) \times 3$

$= 6(3x+1)$

When $x = -1$,

$\frac{dy}{dx} = 6(3(-1)+1)$

$= -12$

Tangent is of the form $y = -12x + c$

Using $(-1, 4)$

$4 = -12(-1) + c$

$c = -8$

$\therefore$ Equation of tangent is $y = -12x - 8$

b $\frac{dy}{dx} = -1(4x)^{-2} \times 4$

$= -\frac{4}{(4x)^2}$

When $x = \frac{1}{4}$,

$\frac{dy}{dx} = -\frac{4}{\left(4 \times \frac{1}{4}\right)^2}$

$= -4$

Tangent is of the form $y = -4x + c$

Using $(0.25, 1)$

$1 = -4(0.25) + c$

$c = 2$

$\therefore$ Equation of the tangent is $y = -4x + 2$

c $\frac{dy}{dx} = 4(3x-5)^3 \times 3$

$= 12(3x-5)^3$

When $x = 2$,

$\frac{dy}{dx} = 12(3(2)-5)^3$

$= 12$

Tangent is of the form $y = 12x + c$

Using $(2, 1)$

$1 = 12(2) + c$

$c = -23$

$\therefore$ Equation of tangent is $y = 12x - 23$

d $\frac{dy}{dx} = \frac{(x-3)(2)-(2x-1)\times 1}{(x-3)^2}$

$= \frac{2x-6-2x+1}{(x-3)^2}$

$= \frac{-5}{(x-3)^2}$

When $x = 4$,

$\frac{dy}{dx} = -\frac{5}{(4-3)^2}$

$= 5$

Tangent is of the form $y = -5x + c$

Using $(4, 7)$

$7 = -5(4) + c$

$c = 27$

$\therefore$ Equation of tangent is $y = -5x + 27$

Question 11

$$\frac{dy}{dx} = (2x-3)(2x)+(x^2-1)(2)$$
$$= 4x^2 - 6x + 2x^2 - 2$$
$$= 6x^2 - 6x - 2$$

$$6x^2 - 6x - 2 = -2$$
$$6x^2 - 6x = 0$$
$$6x(x-1) = 0$$
$$\therefore x = 0,\ x = 1$$

When $x = 0$,

$$y = (2(0)-3)(0^2-1)$$
$$= 3$$

When $x = 1$,

$$y = (2(1)-3)(1^2-1)$$
$$= 0$$

$\therefore$ Points are $(0, 3)$ and $(1, 0)$

Question 12

The y-int (0, 30) gives $d = 30$

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

When $x = 0,\ \frac{dy}{dx} = -1 \ \Rightarrow c = -1$

When $x = 1, \frac{dy}{dx} = -17$

$-17 = 3a + 2b - 1 \qquad \Rightarrow 3a + 2b = -16 \qquad \rightarrow$ Equation 1

When $x = 1, \frac{d^2y}{dx^2} = -10$

$$\frac{d^2y}{dx^2} = 6ax + 2b$$

$-10 = 6a + 2b \qquad \rightarrow$ Equation 2

Solving Equations 1 and 2 simultaneously

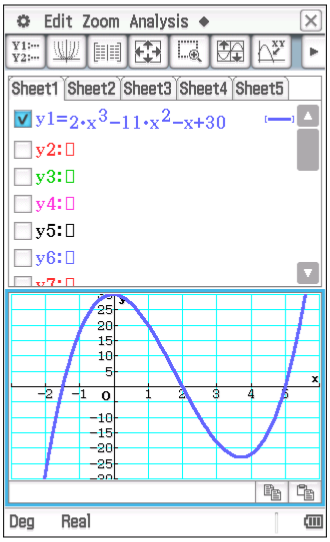
$a = 2, b - 11$

$y = 2x^3 - 11x^2 - x + 30$

$0 = 2x^3 - 11x^2 - x + 30$

By ClassPad $x = -1.5,\ 2,\ 5$

$(-1.5, 0)$, $(2, 0)$ and $(5, 0)$



Question 13

a $\int 20x^3 dx$

$= \frac{20x^4}{4} + c$

$= 5x^4 + c$

b $\int 6x^{\frac{1}{2}} dx$

$= 6x^{\frac{3}{2}} \times \frac{2}{3} + c$

$= 4x^{\frac{3}{2}} + c$

c $\int (x+3)^4 dx$

$= \frac{(x+3)^5}{5} + c$

d $\int (2x+3)^4 dx$

$= \frac{1}{2}\int 2(2x+3)^4 dx$

$= \frac{1}{2} \times \frac{(2x+3)^5}{5} + c$

$= \frac{(2x+3)^5}{10} + c$

e $\int 60x^2(1+x^3)^4 dx$

$= 20\int 3x^2(1+x^3)^4 dx$

$= 20\frac{(1+x^3)^5}{5} + c$

$= 4(1+x^3)^5 + c$

f $\int (1+x^2)^2 dx$

$= \int (1+2x^2+x^4) dx$

$= x + \frac{2x^3}{3} + \frac{x^5}{5} + c$

Question 14

a $\text{Profit} = \text{Revenue} - \text{Cost}$

$= 25.5x - (6000 + 18x)$

$P(x) = 7.5x - 6000$

b $P(x) = 7.5x - 6000 = 0$

$7.5x = 6000$

$x = 800$

c Marginal cost, $C'(x) = 18 \quad \Rightarrow \therefore$ \$18 per unit

Marginal revenue, $R'(x) = 25.5 \quad \Rightarrow \quad \therefore$ \$25.50 per unit

Marginal profit, $P'(x) = 7.5 \quad \Rightarrow \therefore$ \$7.50 per unit

Question 15

$a = 6(t+1)\ \text{m/s}^2$

$v = \int a\,dt$

$\quad = \int 6(t+1)dt$

$\quad = 6\dfrac{(t+1)^2}{2} + c$

$\quad = 3(t+1)^2 + c$

$x = \int v\,dt$

$\quad = \int \left(3(t+1)^2 + c\right)dt$

$\quad = 3\dfrac{(t+1)^3}{3} + ct + d$

$\quad = (t+1)^3 + ct + d$

When $t = 1, x = 3$

$x = (1+1)^3 + c + d$

$3 = 8 + c + d$

$c + d = -5 \qquad \rightarrow \text{Equation 1}$

When $t = 2, x = 19$

$x = (2+1)^3 + 2c + d$

$19 = 27 + 2c + d$

$2c + d = -8 \qquad \rightarrow \text{Equation 2}$

Equation 2 – Equation 1

$2c + d = -8$

$c + d = -5$

$c = -3$

$d = -2$

$\therefore x = (t+1)^3 - 3t - 2$

When $t = 3$,

$x = (3+1)^3 - 3(3) - 2$

$\quad = 53\ \text{m}$

$v = 3(t+1)^2 - 3$

$\quad = 3(3+1)^2 - 3$

$\quad = 45\ \text{m/s}$

Question 16

a $A = \int (2p-1)^3 \, dp$

$= \frac{1}{2}\int 2(2p-1)^3 \, dp$

$= \frac{1}{2} \times \frac{(2p-1)^4}{4} + c$

$= \frac{(2p-1)^4}{8} + c$

When $p = 0$

$0.5 = \frac{(2(0)-1)^4}{8} + c$

$c = \frac{3}{8}$

$A = \frac{\left(2p-1\right)^4}{8} + \frac{3}{8}$

b $A = \int 8p(p^2-1)^3 \, dp$

$= 4\int 2p(p^2-1)^3 \, dp$

$= 4\frac{(p^2-1)^4}{4} + c$

$= (p^2-1)^4 + c$

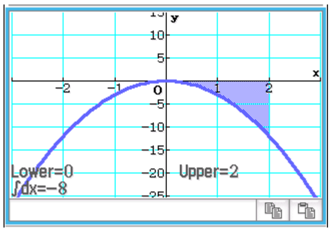
$45 = (2^2-1)^4 + c$

$c = -36$

$\therefore A = (p^2-1)^4 - 36$

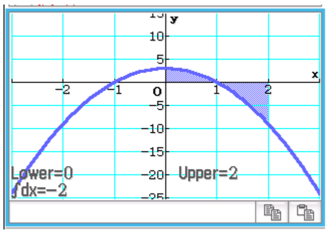
Question 17

a



$$\int_0^2 (-3x^2)\ dx$$
$$= \left[-x^3\right]_0^2$$
$$= (-8-0)$$
$$= -8$$
$$\therefore \text{Area} = 8 \text{ units}^2$$

b



$$3-3x^2 = 0$$
$$3x^2 = 3$$
$$x^2 = 1$$
$$x = \pm 1$$

$$\text{Area} = \int_0^1 (3-3x^2)\ dx + \int_2^1 (3-3x^2)\ dx$$
$$= \left[3x - x^3\right]_0^1 + \left[3x - x^3\right]_2^1$$
$$= \left((3-1)-(0-0)\right) + \left((3-1)-(6-8)\right)$$
$$= 2 + \left(2-(-2)\right)$$
$$= 6 \text{ units}^2$$

Question 18

a

$$\int \left(\frac{3x+1}{\sqrt{x}}\right) dx$$
$$= \int \left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$$
$$= 3x^{\frac{3}{2}} \times \frac{2}{3} + 2\sqrt{x} + c$$
$$= 2x^{\frac{3}{2}} + 2\sqrt{x} + c$$

b

$$\int_4^5 \frac{3x+1}{\sqrt{x}}\ dx$$
$$= \left[2\sqrt{x^3} + 2\sqrt{x}\right]_4^5$$
$$= (2\sqrt{125} + 2\sqrt{5}) - (2\sqrt{64} + 2\sqrt{4})$$
$$= 12\sqrt{5} - 20$$

Question 19

a $y = x^3 - 5x^2 - 6x$

$= x(x^2 - 5x - 6)$

$= x(x-6)(x+1)$

$x^3 - 5x^2 - 6x$ cuts the x-axis at $(-1, 0)$, $(0, 0)$ and $(6, 0)$ and the y-axis at $(0, 0)$.

$y = x^2 - 9x - 10$

$= (x-10)(x+1)$

$x^2 - 9x - 10$ cuts the x-axis at $(-1, 0)$ and $(10, 0)$ and the y-axis at $(0, -10)$.

b $x^3 - 5x^2 - 6x = x^2 - 9x - 10$

$x^3 - 6x^2 + 3x + 10 = 0$

When $x = -1$, $(-1)^3 - 6(-1)^2 + 3(-1) + 10 = 0$

$x = 2$, $2^3 - 6(2)^2 + 3(2) + 10 = 0$

$x = 5$, $5^3 - 6(5)^2 + 3(5) + 10 = 0$

Alternatively,

$x^3 - 6x^2 + 3x + 10 = (x+1)(x^2 + bx + 10)$

$-6x^2 = 1x^2 + bx^2$

$b = -7$

$\therefore x^3 - 6x^2 + 3x + 10 = (x+1)(x^2 - 7x + 10)$

$= (x+1)(x-5)(x-2)$

$(x+1)(x-5)(x-2) = 0$

$x = -1, 2, 5$

When $x = -1$,

$a = (-1)^2 - 9(-1) - 10$

$= 0$

When $x = 2$,

$b = 2^2 - 9(2) - 10$

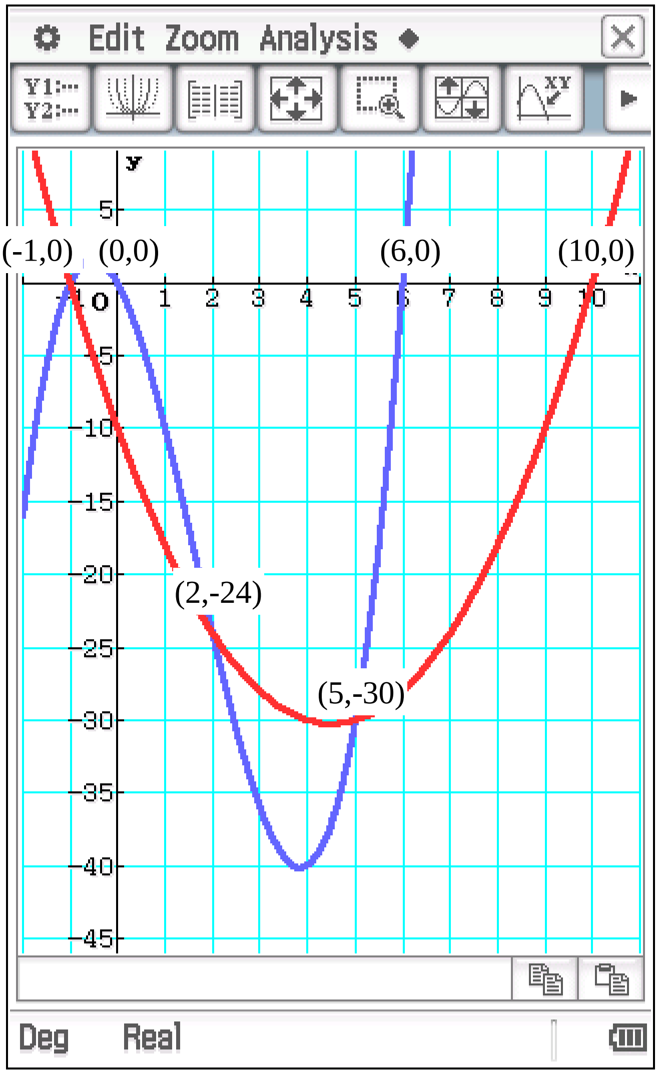
$= -24$

When $x = 5$,

$c = 5^2 - 9(5) - 10$

$= -30$

c

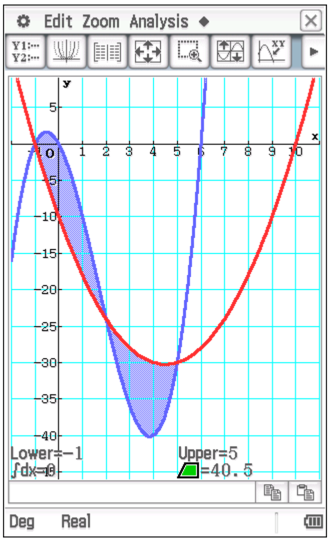


d

$$\int_{-1}^{2}\left((x^3-5x^2-6x)-(x^2-9x-10)\right)dx = 20.25$$

$$\int_{2}^{5}\left((x^2-9x-10)-(x^3-5x^2-6x)\right)dx = 20.25$$

Area enclosed = 40.5 units^2



Question 20

a Given $C(x) = ax^3 - bx^2 + cx$

Average cost per unit = Cost for x units $\div x$

$\dfrac{ax^3 - bx^2 + cx}{x}$

$= ax^2 - bx + c$

To determine when the average cost is a minimum,
find the derivative of the average cost function,
put it equal to zero and solve.

$\dfrac{d}{dx}\left(ax^2 - bx + c\right) = 2ax - b = 0$

$$x = \frac{b}{2a}$$

When $x = \dfrac{b}{2a}$,

Average cost per unit

$= a\left(\dfrac{b}{2a}\right)^2 - b\left(\dfrac{b}{2a}\right) + c$

$= \dfrac{b^2}{4a} - \dfrac{b^2}{2a} + c$

$= \dfrac{b^2 - 2b^2 + 4ac}{4a}$

$= \dfrac{-b^2 + 4ac}{4a}$

Marginal cost $= C'(x) = 3ax^2 - 2bx + c$

When $x = \dfrac{b}{2a}$,

Marginal cost

$= 3a\left(\dfrac{b}{2a}\right)^2 - 2b\left(\dfrac{b}{2a}\right) + c$

$= \dfrac{3b^2}{4a} - \dfrac{2b^2}{2a} + c$

$= \dfrac{3b^2 - 4b^2 + 4ac}{4a}$

$= \dfrac{-b^2 + 4ac}{4a}$

When $x = \dfrac{b}{2a}$, the average cost per unit = marginal cost

b Given $C = f(x)$, then the average cost per unit is $\frac{f(x)}{x}$.

The minimum point for the average cost function:

$$\frac{d}{dx}\left(\frac{f(x)}{x}\right)$$
$$= \frac{xf'(x) - f(x).1}{x^2}$$
$$= \frac{xf'(x) - f(x)}{x}$$

Solving for x when the derivative is equal to zero:

$$\frac{xf'(x) - f(x)}{x} = 0$$
$$xf'(x) - f(x) = 0$$
$$xf'(x) = f(x)$$

Average cost per unit = marginal cost

$$\frac{f(x)}{x} = f'(x)$$
$f(x) = xf'(x)$ as required